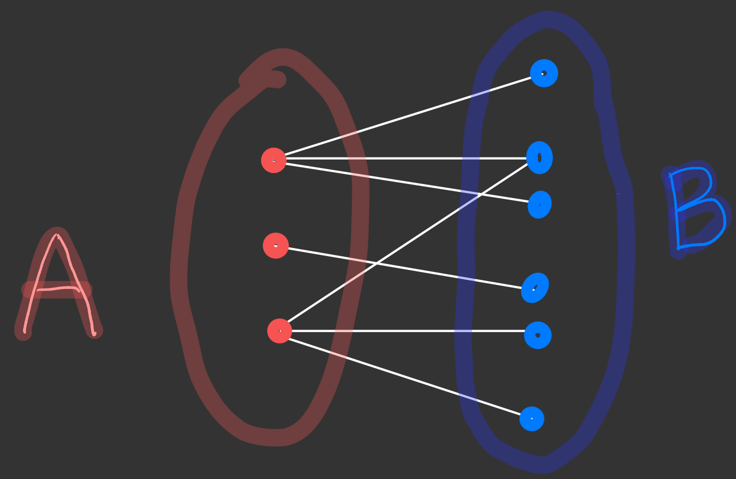
Bipartite graphs

and

Matchings

Definition: A *bipartite graph* is a graph G such that the set of its vertices can be partitioned into two disjoint parts $V(G) = A \sqcup B$ such that every edge of G connects a vertex in A with a vertex in B.

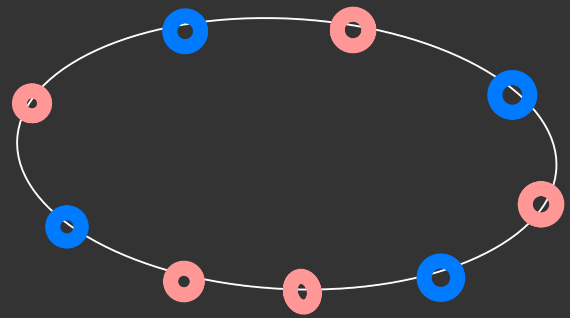
Example:



Lemma: A graph G is bipartite if and only if it does not contain a cycle of odd length.

Proof:

$\Rightarrow$ Suppose that G contains a cycle C of odd length:



Then to subsequent vertices of C belong to the same part (A or B). ↯

$\Leftarrow$ Suppose that G contains no odd cycles. We will show that G is bipartite.

Let $v, w \in V(G)$ be two vertices

Claim: G has no odd cycles $\Rightarrow$ all pathes between v and w has the same length mod 2

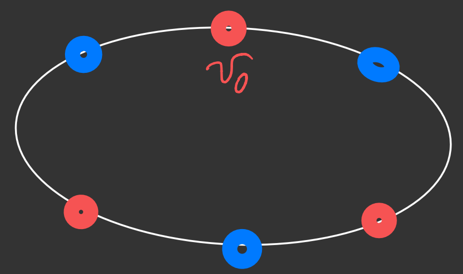
Exercise: prove this claim.

Fix $v_0 \in V(G)$.

We define $A, B \subset V(G)$ as follows:

$A :=$ the set of vertices with even distance from v_0

$B :=$ the set of vertices with odd distance from v_0



No edges of G go between A and A or between B and B.

Is it true that $V(G) = A \sqcup B$?

No.

$A \cup B$ = connected component of G containing v_0

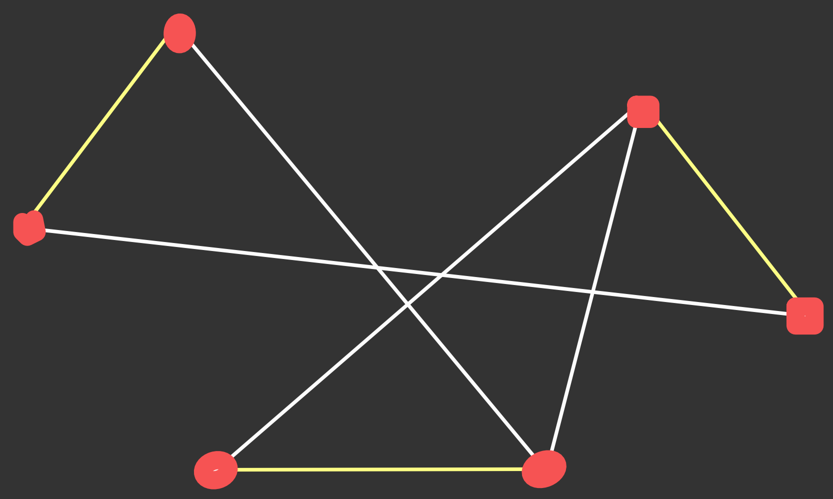
Repeat same procedure for all connected components of G $\blacksquare$

Definition: Let $G = (V, E)$ be a graph.
A subset $E_0 \subseteq E$ of edges such that

$$e \cap f = \emptyset \quad \text{for all} \quad e, f \in E_0$$

is called a *matching* in G.

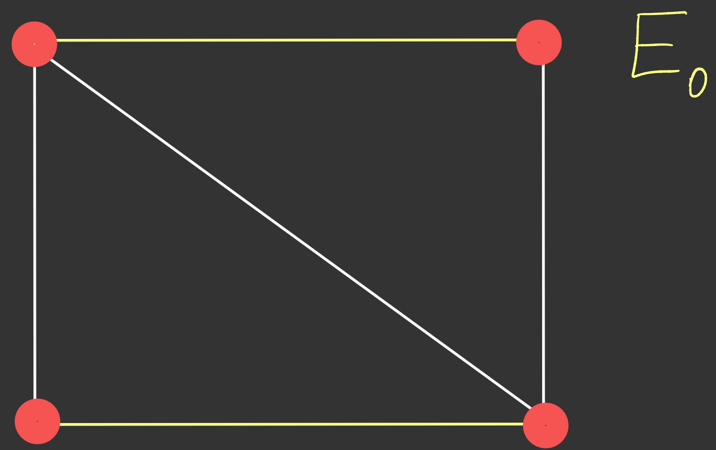
Example:



E_0

Definition: A perfect matching is matching that covers all the vertices of a graph.
That is each vertex is adjoint to exactly one edge of the matching

Example:



Matchings in bipartite graphs

Let G be a bipartite graph such that $V(G) = A \sqcup B$ is a partition.

Does G have a perfect matching?

(Obvious) necessary conditions:

① $|A| = |B|$

② Let $X \subseteq A$ be a subset.

Define

$$Y(X) := \{ y \in B \mid y \text{ is connected to some vertex in } X \}$$

A necessary condition for the existence of a perfect matching:

$$|Y(X)| \geq |X|.$$

Theorem (König-Hall)

Let G be a bipartite graph with two parts A and B.

Suppose that the following two conditions hold:

1. $|A| = |B|$
2. For each set $X \subseteq A$ the subset

$$Y(X) := \{ y \in B \mid y \text{ is connected to some vertex in } X \}$$

satisfies $|Y(X)| \geq |X|$.

Then G has a perfect matching.

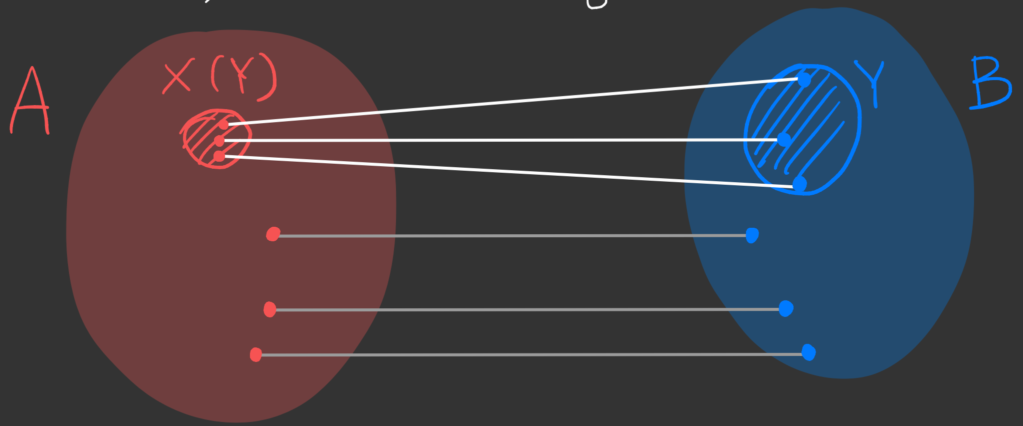
Proof:

We say that a bipartite graph G is "good" if it satisfies conditions ① and ②.

Suppose that a bipartite graph G is good.
Is it true that every subset $Y \subseteq B$ is connected to at least $|Y|$ vertices in A?
In other words, is the condition to be good symmetric with respect to the parts A and B?
We claim that yes.

Indeed, suppose that G is good and there exists a subset $Y \subseteq B$ which is connected to less then $|Y|$ vertices in A.

We denote by $X(Y)$ the set of all vertices connected to Y.



Note that the set $A \setminus X(Y)$ is connected only to the vertices in the set $B \setminus Y$.

By our assumption $|Y| > |X(Y)|$ and therefore

$$|B \setminus Y| < |A \setminus X(Y)|.$$

This contradicts our assumption that G is good.

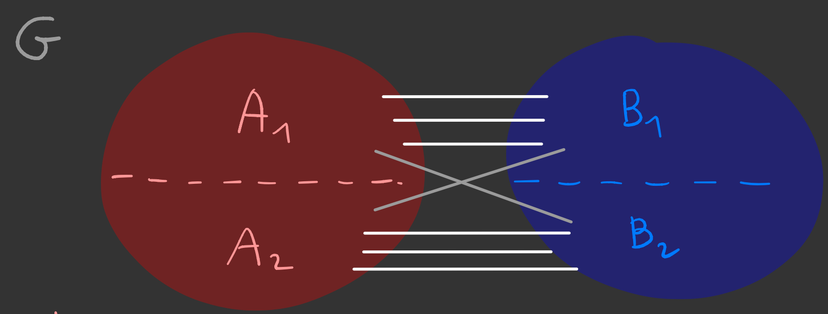
We have to prove that every good bipartite graph G has a perfect matching.

We will prove this by induction on the number of vertices.

For a bipartite graph with 2 vertices the statement is obvious.

We will prove the following:

If a good graph has more then 2 vertices it can be divided into two good graphs:



$A = A_1 \sqcup A_2$ $B = B_1 \sqcup B_2$

$G_1 :=$ subgraph induces by $A_1 \cup B_1$

$G_2 :=$ subgraph induces by $A_2 \cup B_2$

First try:

Let $a \in A$ and $b \in B$ be two vertices connected by an edge.

Set $A_1 := \{a\}$, $B_1 := \{b\}$ and

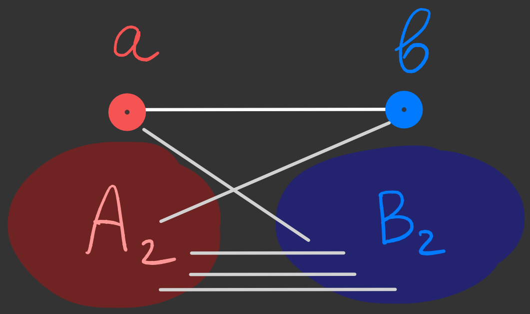
$A_2 := A \setminus \{a\}$ $B_2 := B \setminus \{b\}$.

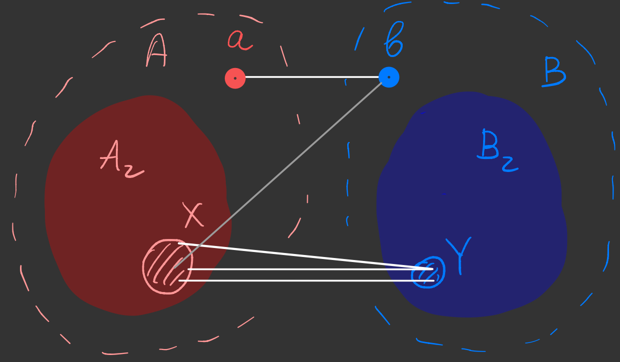
$G_1 :=$ subgraph of G induced by $A_1 \cup B_1$

$G_2 :=$ subgraph of G induced by $A_2 \cup B_2$.

The subgraph G_1 is obviously good.

Is G_2 also good ?





$$\boxed{\begin{array}{l} |X| > |Y| \\ |X| \le |Y \cup \{b\}| \end{array}}$$

$$\Downarrow$$

$$|X| = |Y| + 1$$

Suppose that the graph G_2 is not good and the set A_2 contains a subset X such that the set Y of all vertices in B_2 connected to X contains less then $|X|$ elements.

Since G is good, we know that the number of vertices in B connected to X is exactly $|X|$.

This is possible if and only if $Y(X) = Y \cup \{b\}$.

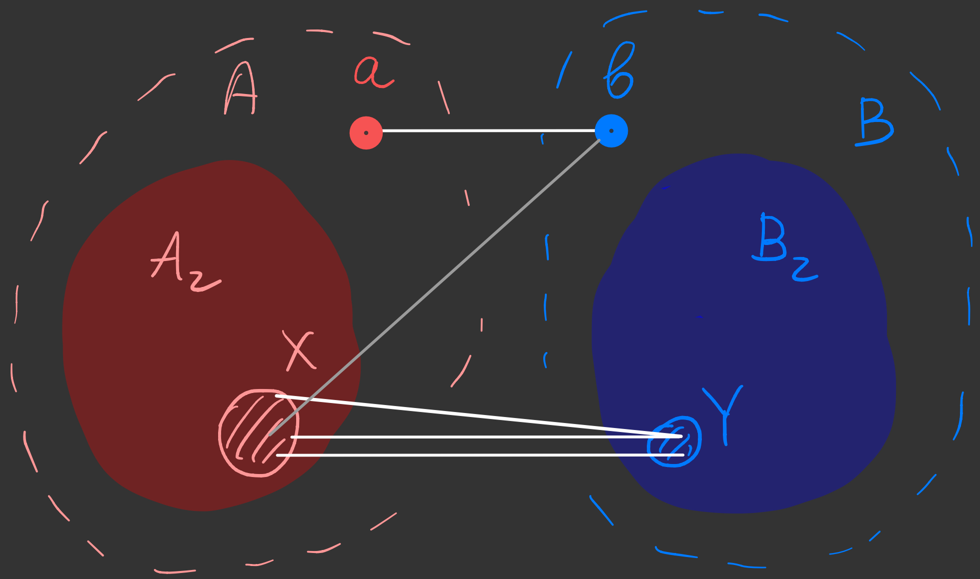
In this case $|X| = |Y \cup \{b\}|$.

We make a second try to divide G.

We define

$$\tilde{A}_1 := X \qquad\qquad \tilde{B}_1 := Y \cup \{b\}$$

X is the smallest subset with property ☼ $\Rightarrow$ the bipartite graph induced on $\tilde{A}_1 \sqcup \tilde{B}_1$ is good.



Is the graph induced by $\tilde{A}_2 := A \setminus X$ and $\tilde{B}_2 := B \setminus Y \setminus \{b\}$ also good?

Vertices of $\tilde{B}_2$ are connected only to the vertices of $\tilde{A}_2$.

Since the graph G is good, every subset $\tilde{Y} \subseteq \tilde{B}_2$ is connected to at least $|\tilde{Y}|$ vertices.

Also $|\tilde{B}_2| = |B| - |Y(X)| = |A| - |X| = |\tilde{A}_2|$.

Therefore, the graph $\tilde{G}_2$ is also good.

Note that $X \subsetneq A$.

This finishes the proof of the statement ( ) and also the proof of the theorem by induction on the number of vertices $\blacksquare$

Theorem: Let G be a non-trivial regular bipartite graph, that is all vertices of G have the same non-zero degree. Then G has a perfect matching.

Proof:

We will show that such a graph G is good.

We will check two conditions of "goodness":

① Both parts have the same number of vertices.

Suppose that G has N edges and each vertex has degree M. Then each bi-part has $\frac{2N}{2M}$ vertices.

② Let us check the second condition.
Suppose that this condition is false.
Then there exists a subset $X \subseteq A$
such that $|Y(X)| < |X|$.
In a subgraph induced by $X \sqcup Y(X)$
the sum of degrees of vertices in X
equals the sum of degrees of vertices in $Y(X)$.
Then $Y(X)$ contains a vertex of degree bigger then M.
This contradicts the regularity of G. $\blacksquare$